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LYAPUNOV EXPONENT OF MAGNETOSPHERIC ACTIVITY FROM AL TIME SERIES

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Abstract. A correlation dimension analysis of the AE index indicates that the magnetosphere behaves as a low-dimensional chaotic system with a dimension close to 4. Similar techniques are used to determine if the system's behavior is due to an intrinsic sensitivity to initial conditions and thus is truly chaotic. The quantity used to measure the sensitivity to initial conditions is the Lyapunov exponent. Its calculation for AL shows that it is nonzero (0.11 ± 0.05 min^{-1}). This gives the exponential rate at which initially similar configurations of the magnetosphere evolve into completely different states. Also predictions of deterministic nonlinear models are expected to deviate from the observed behavior at the same rate.

Introduction

Recent studies [Vassiliadis et al., 1990] of the *correlation dimension* of the magnetosphere based on the analysis of auroral geomagnetic indices have suggested that it behaves as a self-organized system that may be described by a small number of degrees of freedom. Given these findings about global magnetospheric activity two questions arise: first, how strong is the evidence for the small number of variables (or dimensions of the system's state space). Second, given a low number of degrees of freedom, why and when is the behavior of the magnetospheric system irregular, rather than periodic or quasiperiodic?

The low correlation dimension based on analysis of AE and AL data has been recently confirmed by [Roberts, 1991; Shan et al., 1991]. The geomagnetic indices AL, AU, and AE quantify the response of the magnetosphere to solar wind variations as observed in the auroral zone. The currents along field lines that connect the magnetotail to the auroral zone are closed by the ionospheric electrojets. Fluctuations in the westward (eastward) electrojet yield AL (AU), while AE is a measure of both electrojets. The "dimension" of magnetospheric activity is a lower estimate for the system's number of degrees of freedom. More practically the same number of variables and equations would be sufficient to develop a deterministic model of the system. Furthermore, if the number is not an integer it suggests that the evolution of the system in its state space (the space of the variables) is tracing out a fractal pattern of that noninteger dimension. Analysis of AE time series has shown [Vassiliadis et al., 1990] that the correlation dimension of the magnetosphere is fractional (~3.6). The results complement a recent approach for describing the magnetospheric processes in the auroral zone based on simple models of the magnetospheric cavity [Baker et al., 1990; Klimas et al., 1991]. In such models the multitude of driven and intrinsic processes are assumed to be governed by a few dominant degrees of freedom thus allowing for the system approach. While the magnetosphere is modeled by a deterministic dynamical system, its input (corresponding to the solar wind) could be of an irregular, turbulent character.

This letter addresses the second question which is related to the nature of the magnetospheric irregularity. For example the solar wind's erratic variation will affect the activity. Additional to external disturbances though, irregular and unpredictable behavior can be due to *intrinsic deterministic* dynamics. Even systems with a few degrees of freedom (low-dimensional) may exhibit such behavior, apart from the more well-known periodic or quasiperiodic regimes. Because most of these *deterministic chaotic* systems create fractal shapes in their state space, a low, fractional dimension obtained from a space reconstructed from an irregular signal has often indicated that the underlying dynamics is chaotic. However, a fractional dimension does not necessarily imply a chaotic behavior; for instance, colored noise (random phased fluctuations of a power law spectrum) is shown to have low dimension [Osborne and Provenzale, 1989]. The identifying characteristic of a chaotic system, even one of a small number of equations, is related to its "unstable" behavior: a small difference in the initial conditions of two nearby states is exponentially amplified by the strong nonlinear coupling of the variables. For this reason predictions based on a small initial uncertainty (such as an observational error) fail after a finite time, since errors grow rapidly. Computations of the amplification time scale, namely of the Lyapunov exponent which distinguishes between chaotic and random systems, have been made for AL and are presented below.

Lyapunov Exponents from Time Series

Consider a deterministic dynamical system and the time evolution of its n variables given by $\mathbf{x}' = \mathbf{F}(\mathbf{x})$, where $\mathbf{x}$ is an n-dimensional vector. In the case of the magnetosphere $\mathbf{x}$ would contain the global variables that are enough to describe a distinct state or *configuration*, such as the geomagnetic indices, the cross-tail electric field, the size of the magnetosphere, etc. The variables define a state space where the system is represented at each instant by a point and traces out a trajectory, or orbit, during its time evolution. Then the concept of *orbit stability* can be introduced, quantified by Lyapunov exponents and related quantities. An orbit is called *stable* if small variations in initial conditions produce (generally different) orbits which remain in the neighborhood

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of the original one and asymptotically approach it, while an *unstable orbit* is one from which initially nearby trajectories diverge. The divergence will be limited by the maximum length of the state space, but, as soon as the trajectories are close by again, they will "repel" each other. For a globally unstable system this property will characterize almost all of its orbits, any two of which will have a typical behavior as shown in Figure 1. The average rate of divergence can be estimated by the *first Lyapunov exponent* (LE) λ_1 [Wolf et al., 1985; Ruelle, 1989]:

$$\lambda_1 \equiv \lim_{t\to\infty} \frac{1}{t} ln\left(\frac{\Delta x(t)}{\Delta x(0)}\right) = \lim_{t\to\infty} \frac{1}{t} \sum_{i=1}^{t} ln\left(\frac{\Delta x(t_i)}{\Delta x(t_{i-1})}\right) \quad (1)$$

The above definition associates the LE with the direction of highest expansion; in fact there are n LEs altogether, as many as independent directions in the n-dimensional state space.

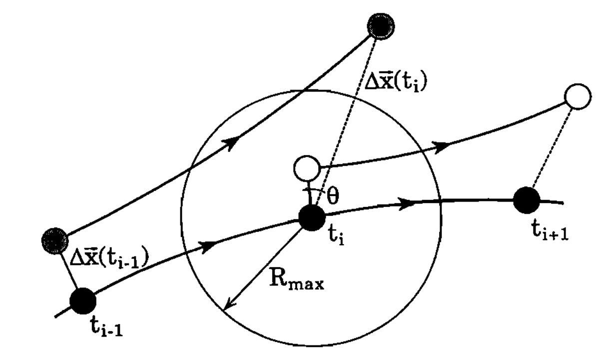

Fig. 1 (after Wolf et al., 1985). For deterministic chaotic systems, trajectories in state space are "unstable": they diverge at an exponential rate given by the Lyapunov exponent. Following a state space trajectory (black circles) it is examined how fast nearby (grey) points diverge by computing the increase of distance from the trajectory. When the distance becomes larger than R_{max} the code looks for a nearer neighbor (white) to replace the former one before the next computation. The angular change θ during replacement should not exceed a parameter value, θ_{max}. The white and grey circles may come from the earlier or later trajectory passes through this region.

A positive (negative) LE gives the rate of expansion (contraction) in the associated direction. LEs are conventionally ranked in decreasing order, with the above equation giving the highest one. The RHS shows that, in effect, a time averaging is performed, the discrete character coming from the finite experimental or numerical time resolution. If all LEs are negative then all trajectories converge to a single point in state space. The presence of at least one positive LE is necessary and sufficient for a deterministic system to be chaotic. On the contrary, most irregular systems of many degrees of freedom (or even infinite, such as random processes) diffuse in state space with their average displacement scaling like a power rather than an exponential of time; for those systems the "LE" as defined in (1) would be zero since the denominator would grow faster and dominate. Thus the existence of at least one positive LE is a distinguishing diagnostic for the chaotic nature of a system.

So far it has been tacitly assumed that all of the variables of the system are subject to measurement. In reality this is rarely the case. To address this issue, embedding theorems have been advanced by Takens [1981]. They state that the "missing" time series can be replaced by time series obtained from the observed ones in a well-defined way. Together with the original one the new time series define the evolution of the system in a "reconstructed" state space. The theorems state the conditions under which the reconstructed state space is equivalent to the original (unavailable) one. It has been shown [Packard et al., 1980] that the reconstruction is successful even from a single time series, and quantities like LEs can be accurately measured in that space.

The LE can be measured using the algorithm of Wolf et al. [1985] for experimental systems whose evolution equations are not known [Mayer-Kress, 1986]. From the time series a trajectory in state space is constructed by the method of delays: if the state space is to have m *embedding dimensions*, m-1 additional time series are needed. The k-th time series is obtained from the original one by a time shift of $(k-1)\tau$, where τ is the *delay*. Its value is a free parameter, but it should be at least as large as the autocorrelation time. The m-dimensional trajectory will have the m time series as its components, so that a point at time t will be defined by **x**(t) = (x(t), x(t+τ), x(t+2τ), ..., x(t+(m-1)τ)). From this it can be seen that given a time series whose length is N the number of points forming the trajectory is going to be N-(m-1)τ. Given a point in that trajectory at time t the following one (at t+1) will be its *image*. The algorithm looks for the nearest neighbor of the point inside a sphere of radius R_{max} (Figure 1). After their mutual distance Δx(t) is recorded the code looks for the images of the two points. The ratio of the new distance Δx(t+1) to the original contributes to the sum (1). The procedure is repeated starting from the point at t+1, until all the points of the trajectory are exhausted. The nearest neighbors should lie as much as possible along the same direction relative to the trajectory. The angle θ between the directions of the new nearest neighbor and the image of the previous one should not exceed a maximum angle θ_{max}. After the appropriate normalization the sum forms the LE. A convergent LE should be independent of the run parameters (length of the time series, embedding dimensions, delay time, etc) for a range of values.

Application to Magnetospheric Activity Time Series

The first (highest) LE from AL time series was computed using the above algorithm. The use of that index rather than AE is based on physical as well as dynamical-systems criteria (e.g. independence of AL from the AU index which enters in calculating AE=AU-AL, smaller sensitivity to magnetospheric currents otherwise unrelated to auroral phenomena; smoother scaling properties of AL with distance in state space [Roberts, 1991]). Initial comparison of several cases showed that AE yields a similar, yet slightly lower (less than 3%) LE than AL.

After test runs on dynamical systems whose LEs have been measured independently the algorithm was applied to the 2.5-min-average AL time series used in [Clauer, 1986] with an autocorrelation time of approximately 3h (Figure 2a). Using a time series of length N=29 k (72.5 k min), an embedding dimension of m=10, and a delay time τ=175 min, the LE was computed and its evolution, as the state space trajectory is scanned, is shown in Figure 2b. Here "time" is measured as the number of trajectory points. The search for nearest

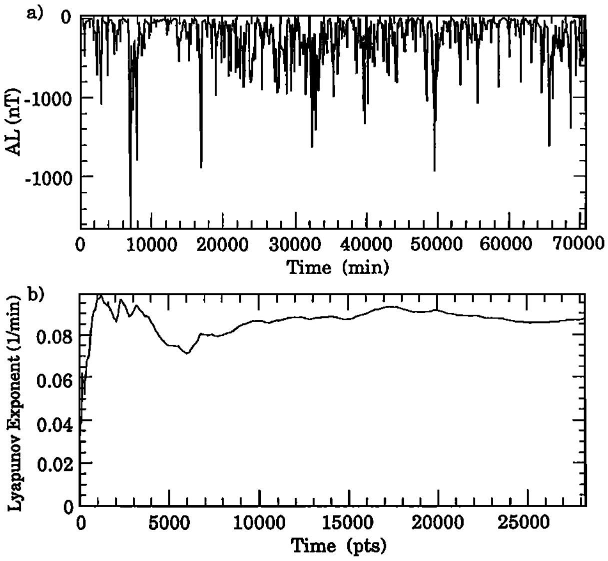

Fig. 2. a) The AL time series (2.5-min-averages). b) Time evolution of the sum in (1) from AL index. Here the length of the time series was N=29k, the embedding dimension m=10, and the time delay τ=3h.

neighbors is made inside a sphere of radius R_{max} = 250 nT. The angular change θ during replacements is less than $\theta_{max} = \pi/5$. This LE shows fluctuations of the order of 0.015 min^{-1} around the value of 0.085 min^{-1}, and corresponds to a time scale of ~12 min.

As the algorithm seeks to determine a dynamical quantity (the LE) from geometrical features of the state space, the radius R_{max} of the sphere used in the search for a nearest neighbor plays an important role. In early runs this variable was set to 35 or 50 nT: when the activity was low (<500 nT) this value was adequate since there were enough points (nearest neighbors) in the sphere. At higher activity levels the sphere was not large enough and contained few neighboring points; statistical fluctuations in the sum (1) were large and the LE was seen to vanish as 1/t. At the next low-activity interval the sum would grow again; in fact plotting it versus 1/t one could obtain its asymptotic value by extrapolation. As R_{max} was increased the evolution of the LE became smoother and the intervals where it decreased disappeared. After discovering this, larger values of R_{max} were used (≥250 nT). The LEs thus obtained from the time series were between 0.06 and 0.17 min^{-1} with fluctuations of ±0.02 min^{-1} or less. Generally the average value of the LE over a run was seen to decrease systematically with the radius.

The embedding dimension m can be chosen from considerations of the number of degrees of freedom of the time series. An embedding dimension equal to $2\nu+1$ unfolds the structure of embedded data of fractal dimension ν [Takens, 1981]. For AL $\nu \approx 4$ so m=9. On the other hand a singular spectrum analysis of AE time series shows that they may be described with 5 independent variables [Sharma et al., 1991]. The LE estimate decreases with m and starts converging around m=9. After that it still drops, at a slower rate though, since the number of trajectory points decreases with embedding dimension. The LE changes less than 5% between m=9 and 10.

The time delay was usually set equal to the first minimum of the autocorrelation function (3h). A plot of the LE versus time delay (Figure 3) shows that longer time delays do not seem to affect the LE significantly; for lower time delays, however, the LE decreases. These runs were shorter than in Figure 2 (N=10 k) and the LE values shown were taken at the end of each run. Initially the length N of the time series was 5–10 k points, but longer time series up to ~30 k points were also tested to ensure proper coverage of the state space in spite of the long delay and high m. It is our experience that time series shorter than 7k points fail to give a consistent estimate. Again, as in correlation dimension calculations it is important that the activity level is homogeneous. This can only be ensured with long enough (≥10k) time series that cover the state space adequately.

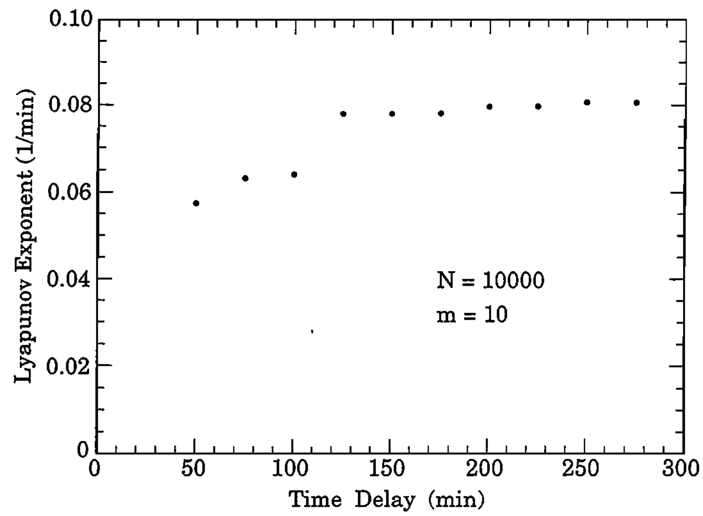

Fig. 3. Variation of AL's Lyapunov exponent with time delay τ. For time delays larger than the autocorrelation time (~3h) the exponent is almost constant. The value of convergence increases slightly with the length of the time series N.

To see if the agreement was only for the cases of high activity of the solar wind (strong driving) a second data set compiled by Bargatze et al. [1985] was used. In this set intervals of AL 2.5-min-averages from the years 1973–74 were arranged in order of increasing activity with quiet periods of two hours separating them. The LE was measured for each one of the two halves of the 40-k-long database. It was found that the differences are small (<10%) and the LE appears independent of the activity level.

The nonzero-LE result is in contrast to what would be obtained from a random process. In fact this is why the sum (1) tends to zero for too low R_{max}: then a point's neighborhood becomes undersampled and nearby points appear to be "randomly" placed. To show that this would be the case with a random signal the following test was performed: the AL time series was Fourier-analyzed, its phases were randomized as would be the case with a random signal of the same spectrum, and then they were Fourier-composed. The random-phased time series had the same autocorrelation time, but its LE was at least three orders of magnitude lower than the original indicating the slower than exponential average separation of nearby points. Increasing the number of points in this randomized time series rapidly decreased the estimate. The diagnostic reached its asymptotic value much faster, sometimes after the first few (50–100) points, in contrast to Figure 2.

Summary and Discussion

The results presented above quantify the irregular magnetospheric behavior observed in auroral zone stations and the coupling of the global magnetosphere to the solar wind parameters. To examine if the AL time series reveals a "chaotic" aspect of the system dynamics (in the sense of an intrinsic strong sensitivity to small variations in initial conditions) and to estimate the associated time scale the Lyapunov exponent was computed. The method is based on reconstruction of the state space from a single variable.

The Lyapunov exponent of AL is in the range 0.06–0.17 min^{-1}. This is close to the independently computed value of Kolmogorov entropy, 0.2 min^{-1} [Vassiliadis et al., 1990] which is defined as the sum of all positive Lyapunov exponents. However, the method for determining the entropy is more sensitive to fluctuations in the state space density. The LE value corresponds to a time scale (~10 min) that shows how fast two initially similar magnetospheric configurations will evolve into different ones. This is a rate of separation for an exponential divergence, as measured in the state space of AL, so the time interval needed for the configurations to be different is a few 1/LE (min). These considerations apply also for the comparison between predictions of a deterministic model and the observed behavior. In this sense this constitutes a limitation for the predictive capabilities of any such model, because initial errors in the determination of magnetospheric activity will be exponentially amplified. This property of the system contributes greatly to its irregular behavior and should be taken into account in magnetospheric modeling.

Additional to this intrinsically unstable nature of the magnetosphere the role of external and other factors has to be examined. In particular, fluctuations in the solar wind and the magnetotail cause disturbances in the magnetosphere which are subsequently amplified. The relation of the level of such fluctuations to the dynamical properties of the system is currently under study. Another important question has to do with the physical mechanism responsible for the exponential divergence and several scenarios based on global models have been proposed [Goertz, 1990; Klimas et al., 1991]. The picture that emerges for the magnetosphere is that of an intrinsically unstable system with an irregular input from its environment.

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